CHAPTER 2

COMPLEX VARIABLES

1. COMPLEX FUNCTIONS►

This chapter focuses primarily on very basic functions, their representations, and properties associated with functions such as limits and continuity.

A complex - valued function *f* of the complex variable z is a rule that assigns to each complex number z in a set **D** one and only one complex number w. We write $w = f(z)$ and call w the image of z under *f*. A simple example of a complex-valued function is given by the formula

$$w = f(z) = z^4 .$$

2■ DOMAIN AND RANGE

DEFINITION 1►

" The set **D** is called the domain of f, and the set of all images $\{w = f(z) : z \in \mathbf{D}\}$ is called the range of f."

We can define the domain to be any set that makes sense for a given rule, so for $w = f(z) = z^4$, we could have the entire complex plane for the domain D, or we might artificially restrict the domain to some set such as $\mathbf{D} = \{z : |z| < 1\}$. Determining the range for a function defined by a formula is not always easy, but we will see

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EXAMPLE 1►

Write $f(z) = z^2$ in the form $f(z) = u(x , y) + j\, v(x , y)$.

SOLUTION

$f(z) = z^2 = (x + j\, y)^2 = (x^2 - y^2) + j\,(2xy)$

$= u(x , y) + j\, v(x , y)$

Equating real and imaginary parts we get

$u(x , y) = x^2 - y^2$, $v(x , y) = 2xy$

EXAMPLE 2►

Express the function $f(z) = \overline{\text{Re}(z)} + z^2 + \text{Im}(z)$, in the form $f(z) = u(x , y) + j\, v(x , y)$

SOLUTION.

Using the elementary properties of complex numbers, it follows that

$$f(z) = (x - jy)(x) + (x + jy)^2 + (y)$$
$$= (2x^2 - y^2 + y) + j(xy) = u(x , y) + j\, v(x , y)$$

so that

$$u(x,y) = 2x^2 - y^2 + y \quad , \quad v(x , y) = x\, y .$$

Examples 1 and 2 show how to find $u(x , y)$ and $v(x , y)$ when a rule for computing f is given. Conversely, if $u(x , y)$ and $v(x , y)$ are two real - valued functions of the real variables x and y, they determine a complex- valued function

$$f(z) = u(x , y) + j\, v(x , y) ,$$

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3 ■ THE POLAR FORM ►

Using $z = r\,e^{j\theta}$ in the expression of a complex function f may be convenient. It gives us the polar representation

$$f(z) = f(r\,e^{j\theta}) = U(r,\theta) + j\,V(r,\theta) \qquad (2)$$

where U and V are real functions of the real variables r and θ

REMARK 1 ►

For a given function f, the functions U and V defined above are different from those used previously in

$$f(z) = f(x + j\,y) = u(x,y) + j\,v(x,y)$$

which used Cartesian coordinates instead of polar coordinates.

EXAMPLE 4 ►

Express $f(z) = z^2$ in polar form.

SOLUTION.

For the polar form, we get

$$f(z) = \left(r\,e^{j\theta}\right)^2 = r^2(\cos 2\theta + j\sin 2\theta)$$
$$= r^2\cos 2\theta + j\,r^2\sin 2\theta = U(r,\theta) + jV(r,\theta)$$

from which we find

$$U(r,\theta) = r^2\cos 2\theta \quad \text{and} \quad V(r,\theta) = r^2\sin 2\theta$$

REMARK 2 ►

Once we have defined u and v for a function f in Cartes form, we must use different symbols if we want to express f in lar form. As is clear from Ex.1 and Ex.4 , the functions u an

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are quite different, as are v and V. Of course, if we are working only in one context, we can use any symbols we choose.

EXAMPLE 5►

Express $f(z) = z^5 + 4z^2 - 6$ in polar form.

SOLUTION

Complete the Solution.....you obtain:

$$U(r,\theta) = r^5\cos 5\theta + 4r^2\cos 2\theta - 6$$

$$V(r,\theta) = r^5 \sin 5\theta + 4r^2\cos 2\theta .$$

4 ■ ELEMENTARY FUNCTIONS►

(i) THE COMPLEX EXPONENTIAL FUNCTION►

Recall that the real exponential function can be represented by the power series $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Thus it is only natural to define the complex exponential e^z also written as exp(z), in the following way:

$$\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \qquad (3)$$

Clearly, this definition agrees with that of the real exponential function when z is a real number. We now show that this complex exponential has two of the key properties associated with its real counterpart and verify the identity

$$e^{j\theta} = \cos\theta + j \sin\theta .$$

If $z = x + jy$, we also see that

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$$\exp(z) = e^{z} = e^{x+jy} = e^{x}\, e^{jy}$$
$$= e^{x} (\cos y + j \sin y) \qquad (4)$$

We now explore some additional properties of $\exp(z) = e^{z}$

Using identity (1.14), we can easily deduce that

1- $e^{z+j2n\pi} = e^{z}$, for all z, provided n is an integer,

2 - $e^{z} = 1$, if and only if $z = j\,2n\pi$, where n is an integer, and

3 - $e^{z_1} = e^{z_2}$, if and only if $z_2 = z_1 + j\,2\,n\,\pi$, for some integer n.

Because Identity (1.14) involves the periodic functions cosy and siny, any two points in the z plane that lie on the same vertical line with their imaginary parts differing by an integral multiple of 2π are mapped onto the same point in the w-plane.

Thus the complex exponential function is periodic with period 2π.

EXAMPLE 1►

For any integer n, the points $z_n = \frac{5}{4} + j\left(\frac{11\pi}{6} + 2n\pi\right)$, aremapped onto a single point

SOLUTION

$$w_n = e^{z_n} = \exp(z_n) \quad , \quad z_n = \frac{5}{4} + j\left(\frac{11\pi}{6} + 2n\pi\right)$$

$$w_n = e^{5/4}\left[\cos\left(\frac{11\pi}{6} + 2n\pi\right) + j\sin\left(\frac{11\pi}{6} + 2n\pi\right)\right]$$

$$w_0 = e^{5/4}\left[\cos(11\pi/6) + j\sin(11\pi/6)\right]$$

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$$w_0 = e^{3/2}\left[\frac{\sqrt{3}}{2} - j\,\frac{1}{2}\right]$$

$$= (3.02272566908) - j\,(1.74517147873)$$

Let's look at the range of the exponential function. If $z = x + j\,y$, we see from identity (4) that e^z can never equal zero, as e^x is never zero, and the cosine and sine functions are never zero at the same point.

Suppose, then, that $w = e^z \neq 0$.

If we write w in its polar form as $w = \rho\, e^{j\varphi}$, identity (4) gives:

$$w = \rho\, e^{j\varphi} = e^x e^{jy} \quad .$$

Using identity (4), we get :

$|\,w\,| = \rho = e^x$, $y = \varphi + 2\,n\,\pi$, where n is an integer.

Therefore, $\rho = |\,e^z\,| = e^x$, and solving these equations for x and y, yields

$$x = \mathrm{Ln}\,(\rho) \text{ and } , \; y = \varphi + 2\,n\,\pi$$

where n is an integer. Thus, for any complex number $w \neq 0$, there are infinitely many complex numbers

$$z = x + j\,y \quad , \quad \text{such that} \quad w = e^z.$$

From the previous equations, we see that the numbers z are

$$z = x + j\,y = \mathrm{Ln}\,(\rho) + j\,(\varphi + 2\,n\,\pi) \qquad (5)$$

where n is an integer.

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(ii) THE COMPLEX LOGARITHM ▶

We showed that, if w is a nonzero complex number, then the equation $w = e^z$, has infinitely many solutions. Because the function $f(z) = e^z$ is a many-to-one function, its inverse (the logarithm) is multivalued.

Definition 2▶ (Multivalued Logarithm)

For $z \neq 0$, we define the function $\ell og(z)$ as the inverse of the exponential function; that is,

$$\ell og(z) = w \quad \text{iff} \quad z = e^w \qquad (6)$$

Thus for any complex number $z \neq 0$, the solutions w to equation (1.16) take the form

$$w = \ell og(z) = Ln \mid z \mid + j\,\theta \qquad (7)$$

where $\theta \in arg(z)$ and $Ln \mid z \mid$ denotes the natural logarithm of the positive number $|z|$. The natural logarithm being used to denote the logarithm in the sense of real variable theory. Because arg(z) is the set

$$arg(z) = \{Arg(z) + 2n\pi : \text{where n is an integer}\} \qquad (8)$$

We can express the set of values comprising $\ell og(z)$ as

$$\ell og(z) = \{Ln\ z| + j\,(Arg(z) + 2n\pi) : \text{where n is an integer}\} \qquad (9)$$

or

$$\ell og(z) = Ln \mid z \mid + j\,arg(z) \qquad (10)$$

Recall that Arg (z) is defined so that for $z \neq 0$, we have

$$-\pi < Arg(z) \leq \pi.$$

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We call any one of the values given in Identities (9) or (10) a logarithm of z.

Notice that the different values of ℓog(z) all have the same real part and that their imaginary parts differ by the amount 2 n π, where n is an integer.

When n = 0, we have a special situation.

DEFINITION 3► (Principal Value of the Logarithm)

For $z \neq 0$, we define the principal value of the logarithm as follows:

$$Log(z) = Ln\ |z| + j\ Arg(z) \quad (11)$$

where $|z| > 0$ and $-\pi < Arg(z) \leq \pi$.

The domain for the function $w = f(z) = Log(z)$ is the set of all nonzero complex numbers in the z - plane, and its range is the horizontal strip $\{ w : -\pi < Im(w) \leq \pi \}$ in the w-plane, and is shown in Figure (2).

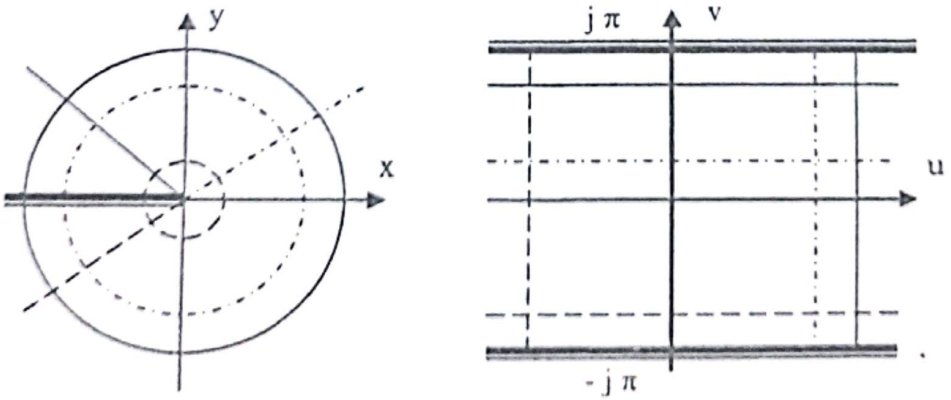


Fig. 2

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Familiar properties of the real – variable logarithm contin
hold but are complicated by the fact that $\ell og(z)$ has infinity
values. Thus , when $z \neq 0$

$$e^{\ell og(z)} = z \ , \text{ but } \quad \ell og(e^{z}) = z + 2\,n\,j\,\pi.$$

Also, Let $z = \alpha\beta$, $\alpha\beta \neq 0$, we get:

$$e^{\ell og(\alpha\beta)} = \alpha\beta = e^{\ell og\alpha}\ e^{\ell og\beta}$$

and hence

$$\ell og(\alpha\beta) = \ell og(\alpha) + \ \ell og(\beta) + 2\,n\,j\,\pi.$$

That is a multiple of $2\pi j$ may be required to validate the above
uation.

To illustrate, consider the following example :

$e^{j\pi} = \cos\pi + j \sin\pi = -1 \quad \Longrightarrow \quad j\ \pi = \ell og(-1)$

Since $\ell og(1) = 0$,

$\ell og(1) = \ell og(-1 \times -1) = \ell og(-1) + \ell og(-1)$

$= j\ \pi + j\ \pi = j\,2\,\pi \neq 0.$

EXAMPLE 1 ▶

Find the values of $\ell og(1 + j)$ and $\ell og(j)$

SOLUTION.

By standard computations, we have

$$\ell og(1+j) = Ln\left|1+j\right| + j\left(Arg(1+j) + 2n\pi\right)$$

$$\ell og(1+j) = Ln\sqrt{2} + j\left(\frac{\pi}{4} + 2n\pi\right)$$, n is an integer, and

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$$\ell og\,(j) = Ln\,|j| + j\,(Arg\,(j) + 2n\pi)$$

$$\ell og\,(j) = 0 + j\left(\frac{\pi}{2} + 2n\pi\right), \text{ n is an integer.}$$

The principal values are

$$\ell og\,(1+j) = Ln\,\sqrt{2} + j\left(\frac{\pi}{4}\right) = \frac{Ln\,2}{2} + j\left(\frac{\pi}{4}\right) ,$$

$$\ell og\,(j) = j\left(\frac{\pi}{2}\right)$$

We now investigate some of the properties of ℓog(z) and Log(z).

1 - exp (ℓog z) = z for all $z \neq 0$. (12)

and

2 - Log (exp z) = z , provided - $\pi < Arg\,(z) \leq \pi$, (13)

When z = x , where x is a positive real number, the principal value of the complex logarithm of z is

$$\ell og\,(x + 0j) = Ln\,x + jArg\,(x) = Ln\,x + j\,0 = Ln\,x \;,\; x > 0$$

Hence Log x is an extension of the real function Ln(x) to the complex case.

C. The Circular Functions :

They are defined as

$$\sin z = \frac{1}{2j}\left(e^{jz} - e^{-jz}\right) \quad , \quad \cos z = \frac{1}{2}\left(e^{jz} + e^{-jz}\right) \quad (14)$$

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They are both analytic functions, and they satisfy most of the relations satisfied by the trigonometric functions sin x ,cos x , e.g.

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

The functions sin z and cos z are periodic with periodicity 2π, but there

are some differences between circular functions and trigonometric functions in the following respect $|\sin x| \le 1 \quad , \quad \forall\, x$

$$|\sin z| = \frac{1}{2}\left|e^{jz} - e^{-jz}\right|$$

when z is on the imaginary axis, i.e., z = j y, we have

$$|\sin z| = \frac{1}{2}\left|e^{-y} - e^{y}\right|$$

$$If \;\; y > 0 \;\; , \;\; \lim_{y\to\infty} |\sin z| = \lim_{y\to\infty} \frac{1}{2}\, e^{y}\left|e^{-2y} - 1\right| = \infty$$

$$If \;\; y < 0 \;\; , \;\; \lim_{y\to-\infty} |\sin z| = \lim_{y\to-\infty} \frac{1}{2}\, e^{-y}\left|1 - e^{2y}\right| = \infty \qquad (15)$$

The other circular functions are defined by means of sin z ,cos z e.g.

$$\tan z = \sin z \,/\, \cos z \,.$$

The other inverse functions are $\sin^{-1} z$, $\cos^{-1} z$, $\tan^{-1} z$,.....

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D. The Hyperbolic Functions :

They are defined as

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right) \quad , \quad \cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right) \quad (16)$$

We can easily deduce the following relations:

$$\sinh(jz) = j\sin z \quad , \quad \cosh(jz) = \cos z$$

$$\sin(jz) = j\sinh z \quad , \quad \cos(jz) = \cosh z$$

E. The Generalized power Function :

Let $w = \ell og\, z = Log\, z + 2\pi k\, j$, $z = re^{j\theta}$.

Hence

$$z = e^{w} = \exp\left(Log\, z + 2\pi k\, j\right)$$

Consider the power function

$$f(z) = z^{n} = e^{nw} = \exp(nw) = \exp\left(n\, Log\, r + jn\theta + 2\pi k\, j\right)$$

(i) If n is ± veinteger , there is only one value for z^n , namely

$$z^{n} = \exp(n\,\ell og\, z) = \exp\left(n\, Log\, r + jn\theta\right) \quad (17)$$

(because exponential function is periodic with period 2 π)

(ii) If n = p/q , q is +ve , then z^n has q - values , namely

$$z^{n} = \exp\left(\frac{p}{q}\,\ell og\, z\right)$$

$$= \exp\left(\frac{p}{q} Log\, r + j\frac{p}{q}\theta + j\frac{2k\,p\,\pi}{q}\right) , \quad k = 0,,1,2,\ldots q-1 \quad (18)$$

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2 ▶ Find all values of $(1+j)^{1-j}$.

3▶ Find all solution of the equation sin z = 3

4 ▶ Find all solution of the equation cash z = -2

5 ▶ Show that (i) $\tan z = \dfrac{\sin 2x + j\sinh 2x}{\cos 2x + j\cosh 2x}$

(ii) $\tanh z = \dfrac{\sinh 2x + j\sin 2x}{\cosh 2x + j\cos 2x}$

6 ▶ If $w = u + jv$ isthe principal value of Ln z , identify the curves of thefamilies v = c , v = k.

7 ▶ If $w = u + jv = \sin^{-1} z$, show that the curves of the family u = C arethe hyperbolas $\dfrac{x^2}{\sin^2 C} - \dfrac{y^2}{\cos^2 C} = 1$. What are the equation of the curves of the family v = k .

DERIVATIVE OF A COMPLEX FUNCTION

From elementary calculus we define the derivative of f(z) at z_0 , written $f'(z_0)$ by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad (1)$$

provided that the limit exists. If it does, we say that the function f(z) is differentiable at z_0. If we write $\Delta z = z - z_0$,then we can express Equation (1.29) in the form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z_0)}{\Delta z} \qquad (2)$$

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If we let $w = f(z)$ and $\Delta w = f(z) - f(z_0)$, then we can write the notation $\frac{dw}{dz}$ for the derivative:

$$f'(z_0) = \frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \qquad (3)$$

EXAMPLE 1 ▶

Use the limit definition to find the derivative of $f(z) = z^3$.

SOLUTION

Using Equation (1), we have

$$f'(z_0) = \lim_{z \to z_0} \frac{(z - z_0)^3}{z - z_0}$$

$$f'(z_0) = \lim_{z \to z_0} \frac{(z - z_0)(z^2 + z z_0 + z_0^2)}{z - z_0}$$

$$f'(z_0) = \lim_{z \to z_0} (z^2 + z z_0 + z_0^2) = z_0^2 + z_0^2 + z_0^2 = 3z_0^2$$

We can drop the subscript on z_0 to obtain $f'(z) = 3z^2$ as a general formula.

■ Pay careful attention to the complex value Δz in Equation (3); the value of the limit must be independent of the manner in which $\Delta z \to 0$.

If we can find two curves that end at z_0 along which $\frac{\Delta w}{\Delta z}$ approaches distinct values, then $\frac{\Delta w}{\Delta z}$ does not have a limit as $\Delta z \to 0$ and $f(z)$ does not have a derivative at z_0. The same observation applies to the limits in Equations (1) and (2).

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EXAMPLE 2►

Show that the function $f(z) = \bar{z}$ is nowhere differentiable.

SOLUTION.

We choose two approaches to the point $z_0 = x_0 + j\,y_0$, and compute limits of the difference quotients. First, we approach $z_0 = x_0 + j\,y_0$ along a line parallel to the x axis by forcing z to be of the form

$z_0 = x + j\,y_0$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad \text{(i)}$$

$$= \lim_{(x+jy_0)\to(x_0+jy_0)} \frac{f(x+jy_0) - f(x_0+jy_0)}{(x+jy_0) - (x_0+jy_0)}$$

$$= \lim_{(x+jy_0)\to(x_0+jy_0)} \frac{(x - j\,y_0) - (x - j\,y_0)}{(x+j\,y_0) - (x_0+j\,y_0)}$$

$$= \lim_{(x+jy_0)\to(x_0+jy_0)} \frac{(x - j\,y_0) - (x_0 - j\,y_0)}{(x - x_0) - j(y_0 - y_0)}$$

$$= \lim_{(x+jy_0)\to(x_0+jy_0)} \frac{(x - x_0)}{(x - x_0)} = 1$$

Next, we approach z_0 along a line parallel to the y axis by forcing z to be of the form $z = x_0 + j\,y$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{(x_0+jy)\to(x_0+jy_0)} \frac{f(x_0+jy) - f(x_0+jy_0)}{(x_0+jy) - (x_0+jy_0)}$$

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$$= \lim_{(x_0+jy)\to(x_0+jy_0)} \frac{(x_0 - j\,y) - (x_0 - j\,y_0)}{(x_0 + j\,y) - (x_0 + j\,y_0)}$$

$$= \lim_{(x_0+jy)\to(x_0+jy_0)} \frac{(x_0 - j\,y) - (x_0 - j\,y_0)}{(x_0 - x_0) - j(y_0 - y_0)}$$

$$= \lim_{(x_0+jy)\to(x_0+jy_0)} \frac{-(y - y_0)}{(y - y_0)} = -1$$

The limits along the two paths are different, so there is no po value for the right side of Equation (i). Therefore $f(z) =$ not differentiable at the point z_0, and since z_0 was arbi $f(z) = \bar{z}$ is nowhere differentiable.

REMARK3 ▶

We showed that $f(z) = \bar{z}$ is continuous for all z. Thus, we a simple example of a function that is continuous everywher differentiable nowhere. Such functions are hard to constru real variables.

We seldom are interested in studying functions that aren't dif tiable, or are differentiable at only a single point. Complex tions that have a derivative at all points in a neighborhood deserve further study.

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DEFINITION 2▶ (Connected Set)

An open set S is a connected set if every pair of points in S can be joined by a polygonal line that lies entirely in S. The open set $|z| < 1$ is connected and so is the annulus $1 < |z| < 2$.

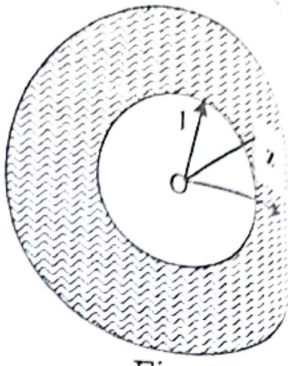


Fig. 2

The set of all points in the plane that do not lie on $|z| = 1$ is an open set which is not connected. This is because we cannot join a point inside the unit circle to a point outside the unit circle without crossi unit circle. The unit circle does not belong to the set.

An open set that is connected is a **domain**. A domain togethe some, none or all of its boundary points is a **region**. A set formed by taking the union of a domain and its boundar **closed region**.

Finally, the set of all points $z (= x + j y)$ such that $y > 0$ is t per half plane. Similarly, for $y < 0$.we have the lower half The conditions $x > 0$ and $x < 0$ define the right half plane a left half plane respectively.

3 ■Analytic Functions▶

Definition 3▶ The complex function f(z) is analytic at the poi z_0 provided there issome $\varepsilon > 0$, such that $f'(z)$ exists for all $z \in D_\varepsilon(z_0)$. In other words, f(z) must bedifferentiable not on at z_0, butalso at all points in some ε - neighborhood of z_0.

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If f(z) is analytic at each point in the region R, then we say that f(z) is analytic on R. Again, we have a special term if f(z) is analytic on the whole complex plane.

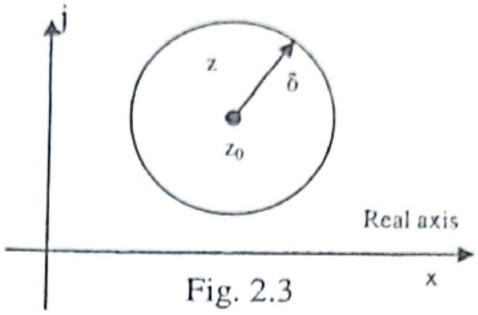


Fig. 2.3

Definition 4► (Entire Function)

"If f(z) is analytic on the whole complex plane then f(z) is said to be entire function".

Definition 5► (Singular Points)

"Points of no analyticity for a function are called singular points. They are important for certain applications in physics and engineering".

Our definition of the derivative for complex functions is formally the same as for real functions and is the natural extension from real variables to complex variables. The basic differentiation formulas are identical to those for real functions, and we obtain the same rules for differentiating powers, sums, products, quotients, and compositions of functions.

4 ■The Rules for Differentiation►

The rules for differentiating functions of a real variable apply equally well for obtaining the derivative of a function of a complex variable. Suppose that f(z) and g(z) are differentiable. Some well-known derivatives are :

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1 ▶ $\frac{d}{dz}C = 0$, where C is a constant,

2 ▶ $\frac{d}{dz}z^n = n\,z^{n-1}$, where n is a positive integer,

3 ▶ $\frac{d}{dz}[C\,f(z)] = C\,f'(z)$,

4 ▶ $\frac{d}{dz}[f(z)+g(z)] = f'(z)+g'(z)$,

5 ▶ $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z)+g(z)f'(z)$,

6 ▶ $\frac{d}{dz}[\frac{f(z)}{g(z)}] = \frac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2}$, $g(z)\neq 0$

7 ▶ $\frac{d}{dz}z^{-n} = \frac{-n}{z^{n+1}}$, $z\neq 0$, n is a positive integer,

8 ▶ $\frac{d}{dz}[F(z)]^n = n[f(z)]^{n-1}f'(z)$, n is a positive integer.

THEOREM 1 ▶

If f(z) is differentiable at z_0 then f(z) is continuous at z_0.

Proof

From Equation (1), we obtain

$$f'(z_0) = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

Using the multiplicative property of limits ,we get

$$\lim_{z\to z_0}[f(z)-f(z_0)] = \lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}\times(z-z_0)$$

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$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \times \lim_{z \to z_0} (z - z_0)$$
$$= f'(z_0) \times 0 = 0$$

This result implies that $\lim_{z \to z_0} f(z) = f(z_0)$, which is equivalent to showing that f(z) is continuous at z_0.

THEOREM 2▶ L'Hôpital's Rule(Without Proof).

"Assume that f(z) and g(z) are both analytic at z_0. If $f(z_0) = 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)} \text{"}. \qquad (4)$$

EXAMPLE 1▶

Find the derivative of $w = z^2$ from first principles.

SOLUTION

$$w = z^2 \qquad \therefore w + \delta w = (z + \delta z)^2 = z^2 + 2z\,\delta z + \delta z^2$$
$$\therefore \delta w = 2z\,\delta z + \delta z^2 \qquad \therefore \frac{\delta w}{\delta z} = 2z + \delta z$$
$$\frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} (2z + \delta z) = 2z$$

EXAMPLE 2▶

Show that the function $f_1 = r^{1/2}[\cos(\theta/2) + j \sin(\theta/2)]$, is discontinuous along the negative real axis.

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SOLUTION

Let $z_0 = r_0 e^{j\pi}$ denote a negative real number. We compu
limit as $z \to z_0$ through the upper half-plane $\{z : \mathrm{Im}(z) > 0$
the limit as $z \to z_0$ through the lower half-plane $\{z : \mathrm{Im}(z) <$
$\}$. In polar coordinates these limits are given by

$$\lim_{(r,\theta)\to(r_0,\pi)} f_1(re^{j\theta}) = \lim_{(r,\theta)\to(r_0,\pi)} r^{1/2}\left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}\right) = j\,r_0^{1/2}$$

and

$$\lim_{(r,\theta)\to(r_0,-\pi)} f_1(re^{j\theta}) = \lim_{(r,\theta)\to(r_0,-\pi)} r^{1/2}\left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}\right) = -j\,r_0^{1/2}$$

As the two limits are distinct, the function $f_1(z)$ is discontinu
z_0.

EXERCISE ► Use L'Hopital's rule to find the following limi

$$\lim_{z\to 1+j} \frac{z^2+z-1-3j}{z^2-2z+2}.$$

ANSWER: $$\lim_{z\to 1+j} \frac{z^2+z-1-3j}{z^2-2z+2} = 1-\frac{3}{2}j\ \#$$

It would be convenient, to have some form of test to see wh
particular function $w = f(z)$ has a derivative $f'(z)$ at
This useful tool is provided by the Cauchy-Riemann equatio

5■CAUCHY – RIEMANN EQUATIONS►

To have a unique derivative, it is necessary that the value of
be independent of the manner in which Δz approaches zer

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cause f(z) is a function of x and y, then

Let $$w = f(z) = u + \mathrm{j}v, \tag{5}$$

We have to establish conditions for $w = f(z)$ to have a derivative at a given point $z = z_0$. Let

$$w = u + \mathrm{j}v \quad \Rightarrow \quad \delta w = \delta u + \mathrm{j}\delta v$$
$$z = x + jy \quad \Rightarrow \quad \delta z = \delta x + \mathrm{j}\delta y$$

Then

$$f'(z) = \frac{\delta w}{\delta z} = \lim_{\delta z \to 0}\left\{\frac{\delta u + \mathrm{j}\delta v}{\delta z}\right\} = \lim_{\substack{\delta x \to 0 \\ \delta x \to 0}} = \left\{\frac{\delta u + \mathrm{j}\delta v}{\delta x + \mathrm{j}\delta y}\right\} \tag{6}$$

(a) Let $\delta x \to 0$, followed by $\delta y \to 0$. Then from (2.6) above,

$$f'(z) = \lim_{\delta y \to 0}\left\{\frac{\delta u + \mathrm{j}\delta v}{\mathrm{j}\delta y}\right\} = \lim_{\delta y \to 0}\left\{\frac{\delta v}{\delta y} - \mathrm{j}\frac{\delta u}{\delta y}\right\} = \frac{\partial v}{\partial y} - \mathrm{j}\frac{\partial u}{\partial y}$$

$$f'(z) = \frac{dw}{dz} = \frac{\partial u}{\partial y} - \mathrm{j}\frac{\partial u}{\partial y} \tag{7}$$

OR

(b) Let $\delta y \to 0$, followed by $\delta x \to 0$. This gives

$$f'(z) = \lim_{\delta x \to 0}\left\{\frac{\delta u - \mathrm{j}\delta v}{\delta x}\right\} = \lim_{\delta x \to 0}\left\{\frac{\delta u}{\delta x} + \mathrm{j}\frac{\delta v}{\delta x}\right\} = \frac{\partial u}{\partial x} - \mathrm{j}\frac{\partial v}{\partial x}$$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + \mathrm{j}\frac{\partial v}{\partial x} \tag{8}$$

If the results (2.7) and (2.8) are to have the same value for $f'(z)$ irrespective of the path chosen for δz to tend to zero, then

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$$\frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - j\frac{\partial u}{\partial y} \qquad (9)$$

Equating real and imaginary parts, this gives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (10)$$

Equations (10) are known as **the Cauchy-Riemann equatio** order that the derivative $f'(z)$ have a unique value which i pendent of the manner in which f(z) approaches zero, it is sary that the real and imaginary parts of f(z) satisfy the C Riemann equations.

Note▶

Because Δx and Δy may approach zero independently, then Δz = Δx + j Δy may approach zero along any path, as is indicated in Fig.4

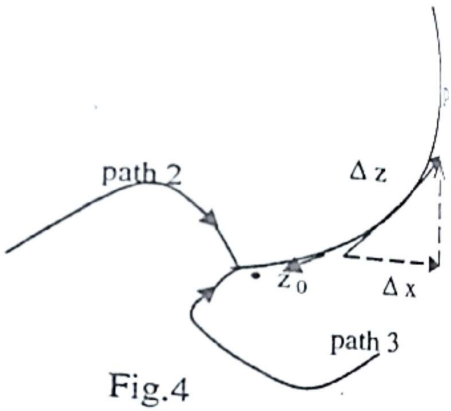


Fig.4

6 ■Cauchy – ReimannIn Polar Form▶

In working with functions that are expressed in polar co nates , it is more convenient to use the polar form of Cauc Riemann equations. The polar form of f(z) is:

$$f(z) = f(r\,e^{j\theta}) = f(r,\theta) = U(r,\theta) + j\,V(r,\theta) \qquad (11)$$

where U(r , θ)is the real part and V(r , θ)is the imaginary part.

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In polar form Δz and $\Delta f(z)$ are :

$$\Delta z = e^{j\theta}\Delta r + j\, r e^{j\theta}\Delta\theta$$

$$\Delta f(z) = \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial \theta}\Delta\theta$$

$$= \left(\frac{\partial U}{\partial r} + j\frac{\partial V}{\partial r}\right)\Delta r + \left(\frac{\partial U}{\partial \theta} + j\frac{\partial V}{\partial \theta}\right)\Delta\theta$$

Substitution of these results into Eq. (2) gives:

$$f'(z) = \lim_{\Delta z \to 0}\frac{\Delta f(z)}{\Delta z}$$

$$= \frac{\left(\frac{\partial U}{\partial r} + j\frac{\partial V}{\partial r}\right)\Delta r + \left(\frac{\partial U}{\partial \theta} + j\frac{\partial V}{\partial \theta}\right)\Delta\theta}{e^{j\theta}\Delta r + j\, r e^{j\theta}\Delta\theta} \qquad (12)$$

For the derivative to exist, it must have the same value for path along which Δr is zero as it does for a path along which $\Delta\theta$ zero. For the case in which Δr is zero, Eq. (12) becomes:

$$f'(z) = \frac{\left(\frac{\partial U}{\partial \theta} + j\frac{\partial V}{\partial \theta}\right)\Delta\theta}{j\, r e^{j\theta}\Delta\theta} = \frac{\frac{\partial V}{\partial \theta} - j\frac{\partial U}{\partial \theta}}{e^{j\theta}} \qquad (i)$$

For a path along which $\Delta\theta$ is zero,

$$f'(z) = \frac{\frac{\partial U}{\partial r} + j\frac{\partial V}{\partial r}}{e^{j\theta}} \qquad (ii)$$

In order that the two preceding equations be equal, it is necessary that the real parts be equal to one another and also that the imaginary parts be equal to one another. Equating the real part gives

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$$\frac{\partial U}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta} \qquad (13)$$

Similarly, equating the imaginary parts gives

$$\frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial U}{\partial \theta} \qquad (14)$$

Equations (13) and (14) are the polar form of the Cauchy-Rie equations.

EXAMPLE 1▶

Show that the function $e^x(\cos y + j \sin y)$,is analytic, find it rivative.

SOLUTION

Let $e^x(\cos y + j \sin y) = u + jv$

$$u = e^x \cos y \quad , \quad v = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \quad , \quad \frac{\partial v}{\partial y} = e^x \cos y$$

$$\frac{\partial v}{\partial x} = e^x \sin y \quad , \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

Since

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The function satisfies the Cauchy-Riemann equations.Hence, $w = f(z) = e^x(\cos y + j \sin y)$ is analytic. To find its deriva

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$$\frac{df}{dz} = f'(z) = \frac{\partial u}{\partial x} + j\frac{\partial v}{\partial x} = e^x \cos y + je^x \sin y$$
$$= e^x(\cos y + j\sin y) = e^x e^{jy} = e^z$$

EXAMPLE 2►

Show that the function f(z) = z², satisfies Cauchy-Riemann equations.

SOLUTION

$$u = x^2 - y^2 \quad , \quad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial v}{\partial x} = 2y = -\frac{\partial u}{\partial y}$$

EXERCISE 1►

Show that the function f(z) = z³, is an analytic function.

EXAMPLE 3►

Test the analyticity of the function sin z, using C-R. equations.

SOLUTION

$$w = \sin z = \sin(x + jy) = \sin x\cos(jy) + \cos x\sin(jy)$$

$$w = u + jv = \sin z = \sin(x + jy)$$

$$= \sin x\cosh y + j\cos x\sinh y$$

$$u = \sin x\cosh y \quad , \quad v = \cos x\sinh y$$

$$\frac{\partial u}{\partial x} = \cos x\cosh y \quad , \quad \frac{\partial v}{\partial y} = \cos x\cosh y$$

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$$\frac{\partial u}{\partial y} = \sin x \sinh y \quad , \quad \frac{\partial v}{\partial x} = -\sin x \sinh y$$

Since

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The function satisfies the Cauchy-Riemann equations.

EXAMPLE 4►

Show that the real and imaginary parts of the function w satisfy the Cauchy equations when z is not zero.

SOLUTION

Let $z = r e^{j\theta} \Longrightarrow w = \log (r e^{j\theta}) = \log r + j\theta$

Using the polar form of C-R equations:

$$\frac{\partial U}{\partial r} = \frac{1}{r}\frac{\partial V}{\partial \theta} \quad , \quad \frac{\partial V}{\partial r} = -\frac{1}{r}\frac{\partial U}{\partial \theta}$$

where for the given function $U = \log r$, $V = \theta$

$$\frac{\partial U}{\partial r} = \frac{1}{r} \quad , \quad \frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{1}{r} \times 1 = \frac{1}{r}$$

$$\frac{\partial V}{\partial r} = 0 = -\frac{1}{r}\frac{\partial U}{\partial \theta}$$

EXAMPLE 5►

Show that the function $w = f(z) = z|z|$, is not analyt

where.

SOLUTION

$$w = f(z) = z|z| = u + jv$$

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rivatives are all continuous and if $\varphi(x, y)$ satisfies Laplace's e tion, then $\varphi(x, y)$ is called a ***harmonic function***.
Taking the partial derivative of Eq. (10) with respect to x and turn and then adding , gives:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (16)$$

Subtracting gives:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad (17)$$

Thus both the real and the imaginary parts u and v of an ana function satisfy Laplace's differential equation. This fact forms basis for the solution of Laplace's equation by potential theory.

THEOREM 3►

Let $f(z) = u(x, y) + j\, v(x, y)$ be an analytic function domain D. Then both $u(x, y)$ and $v(x, y)$ are harmonic f tions on D. In other words, the real and imaginary parts of an lytic function are harmonic.

8 ■CONJUGATE FUNCTIONS►

Definition 6►

If we have a function $u(x, y)$ that is harmonic on the dor D and if we can find another harmonic function $v(x, y)$ such the partial derivatives for u and v satisfy the Cauchy-Rieman uations throughout D. Then we say that $v(x, y)$ is a harmonic jugate of $u(x, y)$ .It then follows that the function

$f(z) = u(x, y) + j\, v(x, y)$ is analytic on D" .

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Further Properties ofConjugate Functions

1 If u (x , y) and v (x , y) are conjugate functions, then the family of curves u(x,y) = $C_1$and the family of curves v (x , y) = C_2, intersect at right angles (orthogonal trajectories).

PROOF

The Eq. u(x , y) = C_1, represents a family of curves the gradient of such curves is given by the gradient (Slope) of the family of curves

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \quad \Rightarrow \quad m_1 = \left(\frac{dy}{dx}\right)_1 = -\frac{u_x}{u_y}$$

The Eq. v (x,y) = C_2, represents a family of curves the gradient of such curves is given by the gradient (Slope) of the family of curves

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 0 \quad \Rightarrow \quad m_2 = \left(\frac{dy}{dx}\right)_2 = -\frac{v_x}{v_y}$$

The product of the two gradients after using Cauchy - Riemann equations is

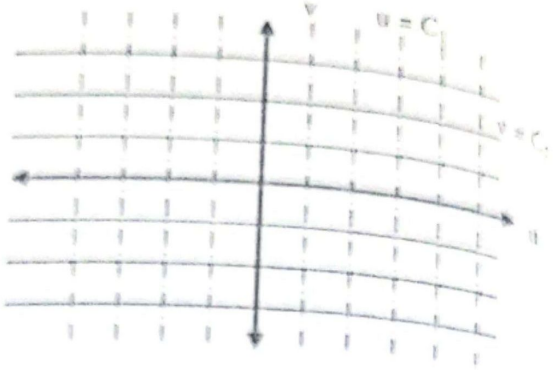
$$m_1 m_2 = \left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2 = \left(-\frac{u_x}{u_y}\right)\left(-\frac{v_x}{v_y}\right) = -1$$

.e. they are orthogonal. (every curve of one family intersect every urve of the other family at right angles) .

XAMPLE 6

onsider the function f (z) = z = x + j y. Then the family u = C_1 x = C_1 , is a family of straight lines parallel to , y- axis. Also, the mily v = C_2 is y = $C_2$is a family of straight lines parallel to x – is as shown in the Figure below.

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It is interesting to consider the application of these equati
logarithmic function.

EXAMPLE 7► Consider the function $w = \ell og\, z$.

SOLUTION

Let $z = r e^{j\theta}$

$w = \ell og\,(r e^{j\theta}) = \text{Log } r + j\theta$

$= U + j V$

Then

$U(r, \theta) = \text{Log } r$ and $V(r, \theta) = \theta$

The family of curves $U(r, \theta) = C_1$ is

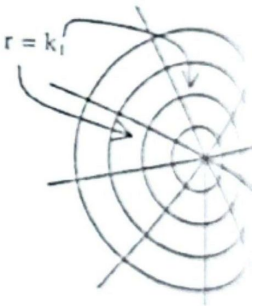


Fig. 2

$\text{Log } r = C_1 \Longrightarrow \quad r = e^{C_1} = k_1$

Which represents a family of concentriccircles of radii k_1

Also. The family of curves

$V(r, \theta) = C_2$ is $\theta = C_2$, which represents a family of stra
passing through the common center of the circles.

Property (ii)▶

The pairs of functions u and v are not chosen at random, and there is an intimate relationship between them.

EXAMPLE

Show that if $\varphi(x, y)$ is harmonic then $\varphi_x - j\varphi_y$, is analytic

SOLUTION

If $\varphi(x, y)$ is harmonic then $\varphi_{xx} + \varphi_{yy} = 0$.

$\varphi_x - j\varphi_y$ is analytic iff the Cauchy-Riemann equations hold.

In this case the C-R equations are

$u = \varphi_x \quad v = -\varphi_y$

$u_x = \varphi_{xx} \quad v_y = -\varphi_{yy} \quad \Rightarrow u_x = v_y \rightarrow \quad \varphi_{xx} = -\varphi_{yy}$

$u_y = \varphi_{yx}, \quad v_x = -\varphi_{xy} \quad \Rightarrow u_y = -v_x \quad \rightarrow \varphi_{yx} = \varphi_{xy}$

The first equation is true because $\varphi(x, y)$ is harmonic and the second equation is true from the usual properties of partial derivatives.

9 ■ Theorem (Construction of a Conjugate)▶

Let u(x,y) be harmonic in an ε -neighborhood of the point (x_0, y_0). Then there exists a conjugate harmonic function v(x , y) defined in this neighborhood such that f(z) = u(x , y) + j v(x , y) is an analytic function.

PROOF

A conjugate harmonic function v will satisfy the Cauchy-Riemann equations $u_x(x\ y) = v_y(x\ ,\ y)$ and $u_y(x\ ,\ y) = -\ v_x(x\ ,\ y)$. Assuming

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that such a function exists, we determine what it would have to look like by using a two-step process. First, we integrate $v_y(x, y)$ (which should equal $u_x(x, y)$) with respect to y and get

$$v(x, y) = \int v_y(x, y)\,dy + C_1(x)$$

$$v(x, y) = \int u_x(x, y)\,dy + C_1(x) \qquad (i)$$

where $C_1(x)$ is a function of x alone that is yet to be determined. Second, we compute $C_1'(x)$ by differentiating both sides of this equation with respect to x and replacing $v_x(x,y)$ with $-u_y(x, y)$ on the left side, which gives

$$v_x(x, y) = \frac{d}{dx}\int u_x(x, y)\,dy + C_1'(x)$$

$$-u_y(x, y) = \frac{d}{dx}\int u_x(x, y)\,dy + C_1'(x) \qquad (2$$

It can be shown that because u is harmonic, all terms except those involving x in the last equation will cancel, revealing a formula for C_1' involving x alone. Elementary integration of the single-variable function C_1' can then be used to discover C_1. We finally observe that the function v so created indeed has the properties we seek. The functions $C(x)$ and $v(x, y)$ are computed with the formulas:

$$C_1(x) = \int\left[-u_y(x, y) - \frac{d}{dx}\left(\int u_x(x, y)\,dy\right)\right]dx$$

$$v(x, y) = \int u_x(x, y)\,dy + C_1(x) \qquad (iii$$

Technically we should always specify the domain of function when defining it. When no such specification is given, it is a

sumed that the domain is the entire complex plane, or the largest set for which the expression defining the function makes sense.

EXAMPLE

If $u(x,y)=e^x \sin y$ find $f(x,y)=u(x,y)+jv(x,y)$

SOLUTION

The Cauchy-Riemann equations are $u_x = v_y$ *and* $u_y = -v_x$

$u(x,y)=e^x \sin y \Rightarrow u_x = e^x \sin y, u_y = e^x \cos y$

$u_{xx} = e^x \sin y, u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = -e^x \sin y + e^x \sin y = 0$

$v_y = u_x = e^x \sin y \Rightarrow v = \int e^x \sin y \, dy + C(x)$

$v = -e^x \cos y + C(x)$

$v_x = -e^x \cos y + \frac{dC(x)}{dx} = -u_y$

Comparing

$\frac{dC(x)}{dx} = 0 \Rightarrow C(x) = k$

$v = -e^x \cos y + k$

$f(x,y)=e^x \sin y - je^x \cos y = -j e^x (\cos y + j \sin y) + jk$

$f(x,y)=-j e^x e^{jy} = -j e^{x+jy} = -j e^z + jk$

EXAMPLE 8►

Show that u(x , y) = $x^2 - y^2$ is a harmonic function and find its harmonic conjugate v(x , y).

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SOLUTION:

$$u_{xx}(x, y) + u_{yy}(x, y) = 2 - 2 = 0,$$

Hence u is a harmonic function for all z.

To find the harmonic conjugate v(x , y) . We use C. R. Eqs.

$$u_x = v_y = 2x \qquad \Rightarrow \quad v = \int 2x\,dy + C_1(x)$$

$$u_y = -v_x = -2y \Rightarrow \quad v_x = 2y$$

From (i)

$$v_x = \int 2\,dy + C_1'(x) = 2y + C_1'(x)$$

Comparing (ii) , (iii) we get:

$$C_1'(x) = 0 \Rightarrow C_1 = const.$$

Therefore v is a harmonic conjugate of u, and the function f(z) by

$$f(z) = (x^2 - y^2) + j(2xy) = (x + jy)^2 = z^2$$

EXAMPLE 9►

Show that $u(x , y) = x y^3 - x^3 y$ is a harmonic function and find harmonic conjugate v(x , y).

Solution. The first partial derivatives are

$u_x(x , y) = y^3 - 3x^2 y$ and $u_{xx}(x , y) = -6xy$

(i) $u_y(x , y) = 3x y^2 - x^3$ and $u_{yy}(x , y) = 6xy$

(ii) Since,

$$u_{xx}(x , y) + u_{yy}(x , y) = -6xy + 6xy = 0$$

Hence, u is harmonic. To construct v, we start with Equation (i)

$$u_x = v_y \qquad \Rightarrow \quad v = \int u_x\,dy + C(x)$$

$$v = \int \left(y^3 - 3x^2 y \right) dy + C(x) = \frac{y^4}{4} - \frac{3}{2} x^2 y^2 + C(x) \qquad \text{(iii)}$$

Differentiating the left and right sides of this equation with respect to x and using $v_x = -u_y$, and Equations (ii) on the left side yields

$$v_x = -3x\, y^2 + C'(x) = -3xy^2 + x^3$$

which implies that $C'(x) = x^3$, then an easy integration yields

$$C(x) = \frac{x^4}{4} + k ,$$

where k is a constant. Now we substitute C (x) and obtain the desired solution

$$v = \frac{y^4}{4} - \frac{3}{2} x^2 y^2 + \frac{x^4}{4} + k .$$

EXERCISE ▶

Show that u = 2x (3 -y) is harmonic and find its harmonic conjugate v. Express the function f(z) = u +j v in terms of z.

Answer:

f(z) = j z^2 +3 z + j C #

EXAMPLE 10 ▶

Let f(z) = u(r , θ) + j v(r , θ) be an analytic function. If $u = -r^3 \sin 3\theta$, then construct the corresponding analytic function f(z) in terms of z.

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SOLUTION

$$u = -r^3 \sin 3\theta$$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta \quad , \quad \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

Using Cauchy-Riemann equations in the polar form.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

$$dv = -\frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right) dr + r\left(\frac{\partial u}{\partial r}\right) d\theta$$

$$dv = -\frac{1}{r}\left(-3r^3 \cos 3\theta\right) dr + r\left(-3r^2 \sin 3\theta\right) d\theta$$

$$= \left(3r^2 \cos 3\theta\right) dr - \left(3r^3 \sin 3\theta\right) d\theta$$

$$v = \int d\left(r^3 \cos 3\theta\right) = r^3 \cos 3\theta + C$$

$$f(z) = u + jv = -r^3 \sin 3\theta + jr^3 \cos 3\theta + jC$$

$$= jr^3 \left(\cos 3\theta + j \sin 3\theta\right) + jC$$

$$= jr^3 e^{j3\theta} + jC = jz^3 + jC$$

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EXERCISE 2

■Determine where each of the following functions fails to be regular, i.e. where singularities occur.

1▶ $w = z^2 - 4$

2▶ $w = \dfrac{z}{z-2}$

3▶ $w = \dfrac{z+5}{z+1}$

4▶ $w = \dfrac{1}{(z-2)(z-3)}$

5▶ $w = z\,\bar{z}$

6▶ $w = \dfrac{x + jy}{x^2 + y^2}$

7▶ If $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$, find f(z).

8▶ If $v = e^x\,(x \sin y + y \sin x)$, find f(z).

9 ▶ If $v(r,\theta) = r^2 \cos 2\theta - r\cos\theta + 2$, find f(z).

10▶ If $v = \log\left(x^2 + y^2\right) + x - 2y$, find f(z).

11▶ If $v = \log\left(r - \dfrac{1}{r}\right)\sin\theta$, find f(z).

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■ Show that the following functions are harmonic and determine the conjugate functions:

12► $u = 4xy - 3x + 2$

13► $u = 2x - x^3 + 3xy^2$

14► $u = \cosh x \cos y$

15► $u = e^{-x}(x \sin y - y \cos x)$

16► $u = x^2 - y^2 + 5x + y - \frac{y}{x^2 + y^2}$.

17► $u = x^2 - y^2 - 2xy - 2x - y - 1$.

18► $u = 3x - 2xy$

CamScanner